MA3111 Complex Analysis I

→ do a presentation to recover missing tutorials
→ in-lecture quizzes (randomly)

- Complex conjugate is distributive over $+ - \times \div$.
- $z\bar{z} = |z|^2$ · On unit circle, $\bar{z} = \frac{1}{z}$
- $|ab| = |a| \cdot |b|$

  $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
- Δ ineq. $|a| + |b| \geq |a \pm b| \geq \big||a| - |b|\big|$
- $e^{i\theta} := \cos\theta + i\sin\theta$ (this is defined like that)
- $x + iy = z = re^{i\theta}$ where $x = r\cos\theta$, $y = r\sin\theta$
- $e^{i\pi} + 1 = 0$
- General power form: $r \cdot \exp[i(\theta + 2\pi n)]$ $(n \in \mathbb{Z})$
- Multiplying in polar form: $r_{z_1 z_2} = r_{z_1} \cdot r_{z_2}$

  $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $(re^{i\theta})^n = r^n e^{i\theta n}$ (by induction on n)
- $\frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$
- $(e^{i\theta})^n = e^{i(\theta n)}$ (where $r=1$) (de Moivre's formula)
- Can derive trigonometric formulas using de Moivre's:

  $(\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)$

  $= \cos(2\theta) + i\sin(2\theta)$

  $\cos(n\theta) = \dfrac{e^{in\theta} + e^{-in\theta}}{2}$

  $\sin(n\theta) = \dfrac{e^{in\theta} - e^{-in\theta}}{2i}$

- n^{th}-roots:

  If $z^n = c = re^{i\theta}$ then $z = (r)^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)}$ for $k = 0, 1, \dots, n-1$

  (we additionally define nth root of zero to be zero)
- $\arg(z)$:= set of possible arguments (infinitely many)
- $\mathrm{Arg}(z)$:= principle argument of z (that is in range $(-\pi, \pi]$)

  ↳ Thm: $\forall z \in \mathbb{C} \setminus \{0\}$, $\mathrm{Arg}(z)$ is unique.
- principle n^{th}-root of $z = c = re^{i\theta}$ is $w := r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n}\right)}$ where θ is the principle argument of z.

  (it is the one closest to zero, on tie prefer positive)
- $\mathbb{C} \cong \mathbb{R}^2$
- $f(z)$ has a limit w_0 at z_0 := $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall z: 0 < |z - z_0| < \delta$, $|f(z) - w_0| < \varepsilon$

  ↖ we ignore the point itself.

Limits results:

If $\lim_{z \to z_0} f(z) = w_1$ and $\lim_{z \to z_0} g(z) = w_2$ then:

$\lim_{z \to z_0} f(z) \pm g(z) = w_1 \pm w_2$

$\lim_{z \to z_0} f(z) g(z) = w_1 w_2$

$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{w_1}{w_2}$ if $g(z), w_2 \neq 0$.

$\lim_{z \to z_0} \bar{z} = \bar{z}_0$ $\lim_{z \to z_0} \overset{\text{polynomial}}{P}(z) = P(z_0)$

Methods of proving $\lim_{z \to z_0} f(z) = w_0$:

(1) use rules for limits and basic limit results

(2) convert the complex function into two functions $\mathbb{R}^2 \to \mathbb{R}$.

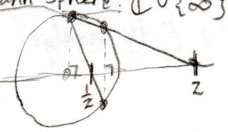
Methods of proving a limit does not exist:

convert the complex function into two functions $\mathbb{R}^2 \to \mathbb{R}$ (or just apply directly on $\mathbb{C} \to \mathbb{C}$)
- and show that if we approach the target point from different directions we get different results.

Continuity

- $f(\cdot)$ is continuous at z_0 := $\lim_{z \to z_0} f(z) = f(z_0)$
- $f \pm g$, $f \cdot g$, $\frac{f}{g}$ inferrable directly from limits.
- f cts $\Rightarrow$ $|f|$ cts.

Riemann Sphere: $\mathbb{C} \cup \{\infty\}$



top	bottom
z	$\frac{1}{z}$
∞	0

- $\lim_{z \to \infty} f(z) := \lim_{s \to 0} f\left(\frac{1}{s}\right)$
- $\lim_{z \to z_0} f(z) = \infty$:= $\lim_{z \to z_0} \frac{1}{f(z)} = 0$

Differentiability := $\lim_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ exists.

Rules: $(f \pm g)' = f' + g'$

$(af)' = a \cdot f'$

$(f \cdot g)' = f'g + fg'$

$\left(\frac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$ (if $g \neq 0$)

Eg. $h(z) = \bar{z}$ is not diffable (since $\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ does not exist)

Necessary condition for diffability:

(1) $\lim_{\Delta x \to 0} \dfrac{f(z_0 + \Delta x) - f(z_0)}{\Delta x}$ exists, $\lim_{i\Delta y \to 0} \dfrac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y}$ exists, and are equal. (where $\Delta x, \Delta y \in \mathbb{R}$).

(i.e. $u_x(x_0, y_0) + iv_x(x_0, y_0) = -iu_y(x_0, y_0) + v_y(x_0, y_0) = f'(z)$

where $z = x + iy$, $z_0 = x_0 + iy_0$) use this formula to calculate $f'(z)$

i.e. Cauchy-Riemann equations: $\begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0) \\ v_x(x_0, y_0) = -u_y(x_0, y_0) \end{cases}$

↑ used to check if something is not diffable.

Sufficient condition for diffability: Cauchy-Riemann equations satisfied, AND u_x, u_y, v_x, v_y cts at (x_0, y_0)

↑ not equivalent

Pf: using MVT.

Topology: Given a set E,

Let $A := \{S \subseteq E\}$

Let $\Sigma \subseteq A$ define a topology on E
must contain ϕ and E.

If $S \in \Sigma$ then we call S "open".

Σ must satisfy: - closure under union (of possibly infinite)
- closure under finite intersection.

Analytic function:

f is analytic at $z_0 := \exists r > 0$ s.t. f is diffable anywhere in $B(z_0, r)$.
open ball.

f is analytic on S $:= \forall x \in S$, f is analytic at x
open set

(Cor: If f is analytic at finite or countably many pts then f is not analytic at any pt.)

Thm: If U is open, then: f is differentiable on $U \Leftrightarrow f$ is analytic on U

Entire function: function that is analytic on whole $\mathbb{C}$.

Prop: If U is connected open set in $\mathbb{C}$ and f analytic on U then: $f'(z) = 0 \Rightarrow f$ is constant.

Harmonic function (on $\mathbb{R}$):

$u(x,y)$ defined on open set S is harmonic if $u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}$ all exist and cts, and $u_{xx}(x_0,y_0) + u_{yy}(x_0,y_0) = 0$ at all $(x_0,y_0) \in S$

Prop: Every analytic function has a harmonic real part and imaginary part, if all partial derivatives exist and are cts.
Pf: By CR eqns.

Harmonic conjugate: If u, v are harmonic functions s.t. $u_x(x_0,y_0) = v_y(x_0,y_0)$ and $u_y(x_0,y_0) = -v_x(x_0,y_0)$ then v is a harmonic conjugate of u.

Thm: v is a harmonic conjugate of $u \Leftrightarrow f(z) = u + iv$ satisfies CR equations $\Leftrightarrow f$ is analytic at any $z \in S$
since u_x, u_y, v_x, v_y cts.

u is a harmonic conjugate of $v \Leftrightarrow \bar{f}$ is analytic at any $z \in S$.

To find a harmonic conjugate: e.g. $u(x,y) = 6xy + e^x \sin y$.

① find u_x and u_y.

② Apply $v_x \underset{(CR)}{:=} -u_y$, so $v(x,y) = \int v_x(x,y)\,dx = \int -u_y(x,y)\,dx = \sim\sim + C(y)$
constant (in terms of y)

③ differentiate $v(x,y)$ to get $v_y = \sim\sim + C'(y)$

④ Use $v_y = u_x$ to solve for $C'(y)$, and hence determine $C(y)$.

Thm conjugate analytic: If f is defined on a connected open set $\subseteq \mathbb{C}$, then

f and $\bar{f}$ both analytic $\Rightarrow f$ is a constant function
$\Updownarrow$
u is a harmonic conjugate of v and v is a harmonic conjugate of u

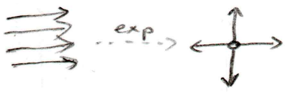
Elementary functions

Exponential: $\mathrm{Exp}(z) = \exp(z) = e^z = e^{\mathrm{Re}(z)}\left(\cos \mathrm{Im}(z) + i \sin \mathrm{Im}(z)\right)$

$= e^x \cos y + i e^x \sin y$ (where $z = x+iy$)

Properties of $\exp(z)$:

- If $z \in \mathbb{R}$ then it reduces to the usual exponent function for $\mathbb{R}$.
- $\frac{d}{dz}\exp(z) = \exp(z)$



- exp is periodic with period $2\pi i$ (i.e. $\exp(z) = \exp(z+2\pi i)$ $\forall z \in \mathbb{C}$)
- $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$
- $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

Logarithm: $\log(z) :=$ inverse of $\exp(z)$. (infinitely many, repeats every 2π)

- If $e^w = re^{i\theta}$ and $w = x+iy$ then $\begin{cases} x = \ln r \\ y = \theta + 2\pi n \ (n \in \mathbb{Z}) \end{cases}$

So $\log(re^{i\theta}) = \ln r + i(\theta + 2\pi n)$ $(n \in \mathbb{Z})$

$\log(z) = \ln|z| + i\arg(z)$

Principle logarithm: $\mathrm{Log}(z) :=$ the value of $\log(z)$ that is in $(-\pi, \pi]$

$\mathrm{Log}(re^{i\theta}) = \ln r + i\mathrm{Arg}(e^{i\theta})$

$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z)$

Properties of $\log(z)$:

$\log(z_1 z_2) = \log(z_1) + \log(z_2)$

Thm of analyticity of Log:

$\mathrm{Log}(z)$ is analytic everwhere except $\{0\}$ and $(-\infty, 0)$

and $\frac{d}{dz}\mathrm{Log}(z) = \frac{1}{z}$

$\{0\}$: $\because \mathrm{Log}(z)$ undefined at 0

$(-\infty,0)$: because $\mathrm{Arg}(z)$ is discontinuous there (principle arg changes)

Different log: $\log(z)$ with $\alpha < \arg(z) \leq \alpha + 2\pi$:= the value of $\log(z)$ that is in $(\alpha, \alpha+2\pi]$

↳ might want to use, for choosing where the discontinuity is.

cut complex plane $:= \mathbb{C} \setminus \left(\{z \in \mathbb{C} \mid \arg z = \alpha\} \cup \{0\}\right)$

Power functions: $z^c := e^{c \log z}$

- If $c \in \mathbb{Z}$, $z^c = e^{c \cdot \mathrm{Log}\, z}$ ← single unique value
- If $c = \frac{m}{n} \in \mathbb{Q}$ $(m, n \in \mathbb{Z}, \gcd(m,n)=1)$, $z^c = e^{\frac{m}{n}\mathrm{Log}(z)} \cdot \underbrace{e^{\frac{2mk\pi}{n} i}}_{\text{has } n \text{ different values}}$; $k \in \mathbb{Z}$

Principal value of z^c $= \mathrm{P.V.}\ z^c = e^{c \,\mathrm{Log}\, z}$

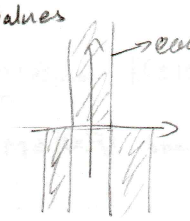
Trigonometric Functions: $\cos z := \frac{e^{iz} + e^{-iz}}{2}$

$\sin z := \frac{e^{iz} - e^{-iz}}{2i}$

$\cos(z) = \cosh(iz)$, $\sin(z) = -i \sinh(iz)$

Hyperbolic functions: $\cosh z := \frac{e^z + e^{-z}}{2}$

$\sinh z := \frac{e^z - e^{-z}}{2}$



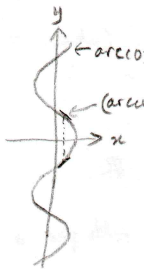
each strip maps to $\mathbb{C} \setminus \{0\}$ under cos.

- all usual algebraic identities for trigonometric functions still hold.

Differentiation: (as per normal like $\mathbb{R}$)

- $\left(\sum_{i=0}^{N} a_i z^i\right)' = \sum_{i=1}^{N} i a_i z^{i-1}$
- $(e^z)' = e^z$
- $(\operatorname{Log} z)' = \frac{1}{z}$
- $(z^c)' = c z^{c-1}$ (where $c \in \mathbb{Z}$)
- $(\sin z)' = \cos z$
- $(\cos z)' = -\sin z$
- $(\sinh z)' = \cosh z$
- $(\cosh z)' = \sinh z$
- $(\arcsin z)' = \frac{1}{(1-z^2)^{\frac{1}{2}}}$ (two values)
- $(\arccos z)' = \frac{-i}{(z^2-1)^{\frac{1}{2}}}$ (two values)
- $(\arctan z)' = \frac{1}{1+z^2}$ (one value)

- $\cosh^2 z - \sinh^2 z = 1$
- $\arccos z = -i \log(z + (z^2-1)^{\frac{1}{2}})$ (infinitely many values)



- $\arcsin z = -i \log(iz + (1-z^2)^{\frac{1}{2}})$

Integration:

Curve: $\gamma : [a,b] \to \mathbb{C}$ (Note: curve also measures the "time" – two different curves can have the same track)

- **simple**: no self-intersection
- **closed**: $\gamma(a) = \gamma(b)$
- **simple closed**: $\gamma(a) = \gamma(b)$ and $\forall c \in (a,b), \nexists d \in [a,b] \setminus \{c\}$ s.t. $\gamma(c) = \gamma(d)$
- **smooth**: γ' is defined and cts (where $\gamma'(t) := x'(t) + iy'(t)$, $\gamma(t) =: x(t) + iy(t)$, $\gamma'(t)$ cts $:=$ $x'(t)$ and $y'(t)$ both cts)

Formulas:

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$$\int_a^b u(t) + iv(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

$$\int_{\text{unit circle}} z^n\,dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

Note: If two curves γ_1 and γ_2 have the same track, then $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz \quad \forall z \in \mathbb{C}$.

Thm: Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth curve, and $\phi : [c,d] \to [a,b]$ s.t. ϕ' exists and cts on $[c,d]$ and $\phi(c) = a$ and $\phi(d) = b$, and $\alpha(t) := \gamma(\phi(t))$, then for any cts f on α (i.e. on γ), $\int_\gamma f(z)\,dz = \int_\alpha f(z)\,dz$.

Def: If $\gamma : [a,b] \to \mathbb{C}$ then
$-\gamma : [-b,-a] \to \mathbb{C}$ where $(-\gamma)(t) := \gamma(-t)$, and $\int_\gamma f(z)\,dz = -\int_{-\gamma} f(z)\,dz$.

Thm: When integrating on a path in $\mathbb{C}$:
$$\left|\int_a^b f(z)\,dz\right| \le \int_a^b |f(z)|\,dz \le \max|f(z)| \cdot \underbrace{\text{length}(a,b)}_{\text{dist. on path}}$$

Thm: ML-inequality: Suppose f cts on γ, and $|f(z)| \le M$. Then $\left|\int_\gamma f(z)\,dz\right| \le M \cdot \underbrace{L}_{\text{length of } \gamma}$

can be proved using $\int_a^b |\gamma'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$

Contour: multiple curves joint at endpoints.

<u>Antiderivative</u>: If $f, F : S \to \mathbb{C}$ ($S \subseteq \mathbb{C}$) and F is analytic and $F' = f$ then F is an <u>antiderivative</u> of f.

<u>Rmk</u>: If S open & connected, then all antiderivatives of f differ by a constant.

<u>Fundamental Thm of Calculus</u>: Suppose f has an antiderivative F on (open & connected) domain D, and $z_1, z_2 \in D$. Then regardless of the contour $\gamma : [z_1, z_2] \to \mathbb{C}$, $\int_\gamma f(z)\,dz = F(z_2) - F(z_1)$

<u>Cor</u>: If γ is a closed contour, then $\int_\gamma f(z)\,dz = 0$.

<u>Converse Thm</u>: If all contour integrals on $f : D \to \mathbb{C}$ ($D \subseteq \mathbb{C}$ domain) are independent of the contour, then f has an antiderivative.
i.e. $\forall z_1, z_2 \in D, \exists k \in \mathbb{C}$ s.t. $\forall \gamma : [z_1, z_2] \to \mathbb{C}$, $\int_\gamma f(z)\,dz = k$

<u>Note</u>: When integrating a function with a hole in its domain (e.g. $f(z) = \frac{1}{z}$), paths not topologically equivalent may not have equivalent path integral even if f is analytic.

<u>Cauchy-Goursat Thm</u>: If f is analytic at all points interior to and on a simple closed contour γ, then $\int_\gamma f(z)\,dz = 0$ → <u>Pf</u>: using Green's thm.

<u>Jordan curve thm</u>: Every simple closed curve has an interior (i.e. bounded) part and an exterior (i.e. unbounded) part.

<u>Cor. of C-G thm</u>: Given two positively oriented simple closed contours γ_1, γ_2, if f is analytic on the closed region between γ_1 and γ_2, then $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$.

<u>Simply connected domain</u>: An open and connected set with no "holes".

<u>Cor. of C-G thm</u>: If f is analytic on a simply connected domain D, then for any (not necessarily simple) closed contour γ in D, $\int_\gamma f(z)\,dz = 0$.

<u>Cauchy integral formula</u>: If γ is a positively oriented simple closed contour and f is analytic <u>on</u> and <u>in</u> γ. Then for any z_0 <u>in</u> γ,

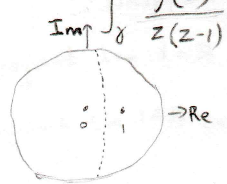
$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,dz.$$

<u>Pf</u>: do the integral on a small circle around z_0.

<u>Generalisation</u>: If γ is closed (not necessarily simple) and f is analytic on and in γ, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,dz = f(z_0)\cdot(\text{winding number of } \gamma \text{ around } z_0)$$

<u>Method of splitting</u>:



$$\int_\gamma \frac{f(z)}{z(z-1)}\,dz = \int_{\gamma_1} \frac{f(z)}{z(z-1)}\,dz + \int_{\gamma_2} \frac{f(z)}{z(z-1)}\,dz$$
$$= 2\pi i\left(f(0)/(0-1)\right) + 2\pi i\left(f(1)/1\right)$$
$$= 2\pi i\,(f(1) - f(0))$$

<u>Generalisation for derivatives</u>:

If γ : positively oriented simple closed contour
f : analytic <u>on</u> and <u>in</u> γ.
Then for any z_0 <u>in</u> γ, for any $n = 0, 1, 2, 3, \ldots$

$$\frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\,dz = f^{(n)}(z_0)$$

<u>Pf</u>: Differentiate the original C.I.F.

<u>Morera's thm</u>. If f cts on domain D and for every closed contour $\gamma \in D$, $\int_\gamma f(z)\,dz = 0$, then f analytic in D.

<u>Analyticity and Antiderivative</u>

If f is analytic in an open & simply connected set D, then f has an antiderivative on D.

<u>Analyticity and Derivative</u>

· If f is analytic in an open & connected set D, then:
(1) f has derivatives of any order.
(2) If $f(z) = u(x,y) + iv(x,y)$ then u & v have partial derivatives of any order.

<u>Cauchy inequality</u>:

Let $f(z)$ be analytic on & in the circle γ with radius $R > 0$ centred at z_0.
Let $M_R := \max_{z \in \gamma} |f(z)|$. Then $|f^{(n)}(z_0)| \le \frac{n! \cdot M_R}{R^n}$ for any $n \in \mathbb{Z}^+$

<u>Liouville theorem</u>: If f is entire & bounded, then f is constant. <u>Pf</u>: corollary of Cauchy's inequality.

Cor: If f is entire and $Re(f(z)) \geq 0 \ \forall z \in \mathbb{C}$, then f is constant.

PF: consider $g(z) = e^{-f(z)}$.

Fundamental Thm of Algebra: Any polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ has at least one complex root $z_1 \in \mathbb{C}$ (s.t. $f(z_1) = 0$)

PF: If not, then $\frac{1}{p(z)}$ is entire. But we can show that $\frac{1}{p(z)}$ is bounded. So apply Liouville to show $\frac{1}{p(z)}$ is constant.

$\mathbb{C}$ is complete (so Cauchy $\Leftrightarrow$ convergent)

Convergence of geometric series: $\sum_{n=1}^{\infty} a z^{n-1} \begin{cases} \text{converges to } \frac{a}{1-z} & \text{if } |z| < 1 \\ \text{diverges} & \text{otherwise.} \end{cases}$

Continuity of uniform convergence: If $(f_n)_{n \in \mathbb{N}}$ all cts and $f_n \xrightarrow{unif.} f$, then f cts.

Integration & uniform convergence: If $(f_n)_{n \in \mathbb{N}}$ all cts and $f_n \xrightarrow{unif} f$, then $\lim_{n \to \infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz$.

Differentiation & uniform convergence: If $(f_n)_{n \in \mathbb{N}}$ are analytic functions on an open set D and $f_n \xrightarrow{unif} f$ on D then:
- f is analytic on D
- $\lim_{n \to \infty} f_n'(z) = f'(z)$ for all $z \in D$

Weierstrass M-test: Consider a series of functions $\sum_{k=1}^{\infty} f_k$ on set $D \subseteq \mathbb{C}$.
Suppose: (1) $|f_k(z)| \leq M_k$ for all $z \in D$, $k \in \mathbb{N}$.
(2) $\sum_{k=1}^{\infty} M_k$ converges
Then $\sum_{k=1}^{\infty} f_k$ converges uniformly on D.

Cor: In the power series $f_n = a_n(z-z_0)^n$, $n \in \mathbb{N} \cup \{0\}$, $\sum f_n$ converges uniformly on $\{z \in \mathbb{C} \mid |z-z_0| \leq \rho\}$ for any $\rho < (\limsup |a_n|^{\frac{1}{n}})^{-1} =: R$
and $\sum f_n$ converges ptwise on $\{z \in \mathbb{C} \mid |z-z_0| < R\}$

Thm: Let $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ be a power series with radius of convergence R. Then:
(1) $S(z) := \sum_{k=0}^{\infty} a_k(z-z_0)^k$ is analytic on $B(z_0, R)$. ← open ball
(2) $S'(z) = \frac{d}{dt} \sum_{k=0}^{\infty} a_k (z-z_0)^k = \sum_{k=1}^{\infty} k a_k (z-z_0)^{k-1}$
(3) If $\gamma \subset B(z_0, R)$ then
$$\int_\gamma g(z) S(z) dz = \int_\gamma g(z) \sum_{k=0}^{\infty} a_k (z-z_0)^k dz = \sum_{k=0}^{\infty} a_k \int_\gamma g(z)(z-z_0)^k dz$$
i.e. can swap the integral with the sum.

Special cases on the circle of convergence:
(1) $\sum_{k=1}^{\infty} z^k$ has $R=1$ and diverges everywhere on the circle of convergence
(2) $\sum_{k=1}^{\infty} \frac{1}{k} z^k$ has $R=1$ and diverges at $z=1$ but converges everywhere else on the circle of convergence
(3) $\sum_{k=1}^{\infty} \frac{1}{k^2} z^k$ has $R=1$ and converges everywhere on the circle of convergence

Taylor theorem: Suppose $f(z)$ is analytic in $B(z_0, R)$. Then $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$ for any $z \in B(z_0, R)$.

Cor: radius of convergence $= \min\{|z-z_0| : f \text{ is nonanalytic at } z\}$

Computation of Power Series:
- $e^z = \frac{z^0}{0!} + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$
- $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{z^1}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$
- $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots \quad (|z| < 1)$
- $\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{z^0}{0!} - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$
- $\frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)' = 1 + 2z + 3z^2 + 4z^3 + \dots \quad (|z| < 1)$
- $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots \quad (|z| < 1)$

Methods to find Maclaurin series:
E.g. $f(z) = \frac{1}{1-2z^2}$ at $z_0 = 0$

E.g. $\frac{1}{z}$ at $z_0 = i$: $\frac{1}{z} = \frac{1}{i} \cdot \frac{1}{1-i(z-i)} = \frac{1}{i}(\dots)$

E.g. $\frac{e^z}{1-2z} = \frac{1+z+\frac{z^2}{2}+\dots}{1-2z}$ (use long division, or let $\frac{1}{1-2z} = 1 + 2z + \dots$ and multiply.

(1) Let $w = 2z^2$, find expansion of $\frac{1}{1-w}$, and substitute back (substitute validity bounds too)
(2) Partial fractions: $f(z) = \frac{1}{2}\left(\frac{1}{1+i\sqrt{2}z} + \frac{1}{1-i\sqrt{2}z}\right)$, expand separately and combine bounds.

- $\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \dots$

Annulus: $\operatorname{Ann}(z_0, R_1, R_2) := \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$

$\overline{\operatorname{Ann}(z_0, R_1, R_2)} := \{z \in \mathbb{C} \mid R_1 \le |z - z_0| \le R_2\}$

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}}_{\text{principal part (tends to 0 as } z\to\infty)}$$

$||$

Laurent's thm: Suppose $f(z)$ is analytic in $\operatorname{Ann}(z_0,R_1,R_2)$. Then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ where $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{(s-z_0)^{n+1}}\, ds$

where γ is any positively oriented simple closed contour around z_0 and lying inside $\operatorname{Ann}(z_0,R_1,R_2)$.

Methods to find Laurent series:

E.g. Find Laurent series of $f(z) = \frac{1}{z}$ for $3 < |z-3| < \infty$.

Sol: $f(z) = \frac{1}{(z-3)+3} = \frac{1}{z-3} \cdot \frac{1}{1+\frac{3}{z-3}} = \frac{1}{z-3}\left(\sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{z-3}\right)^n\right) = \sum_{n=0}^{\infty} (-3)^n (z-3)^{-n-1} = \sum_{n=1}^{\infty} (-3)^{n-1} (z-3)^{-n}$ $\quad (3 < |z-3| < \infty)$

since $\left|\frac{3}{z-3}\right| < 1$, can apply Taylor series of $\frac{1}{1+z}$

E.g. Find Laurent series of $f(z) = \frac{3z+5}{(z+1)(z+2)}$ for $1 < |z| < 2$.

Sol: $f(z) = \frac{2}{z+1} + \frac{1}{z+2}$ (partial fractions). $\quad \frac{2}{z+1} = \frac{2}{z}\left(\frac{1}{1+\frac{1}{z}}\right) = \frac{2}{z}\sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2}{z^{n+1}}$. $\quad \frac{1}{z+2} = \frac{1}{2}\left(\frac{1}{1+\frac{z}{2}}\right) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n$

since $\left|\frac{1}{z}\right| < 1$, can apply Taylor series of $\frac{1}{1+z}$ $\quad$ since $\left|\frac{z}{2}\right| < 1$, can apply Taylor series of $\frac{1}{1+z}$

$\therefore f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n \quad (1 < |z| < 2)$

E.g. Find $\int_{|z|=\frac{3}{2}} \frac{3z+5}{z^4 (z+1)(z+2)}\, dz$.

Sol: Use Laurent's thm for a_3: $\quad a_3 \cdot 2\pi i = \int_{|z|=\frac{3}{2}} \frac{3z+5}{z^4(z+1)(z+2)}\, dz$. $\quad a_3 \cdot 2\pi i = \frac{-1}{2^4} \cdot 2\pi i = -\frac{\pi i}{8}$

Isolated singular point: a nonanalytic point with only analytic points in its neighbourhood (singular point: a nonanalytic point with some analytic point in its neighbourhood)

E.g. Find $\int_{|z|=2} f(z)\, dz$: Find a_{-1} of the Laurent series of $f(z)$.

Residue (Method 4): If z_0 is an isolated singleton point then $a_{-1} = b_1 = \frac{1}{2\pi i}\int_\gamma f(z)\, dz$ is called the residue of f at z_0 $=: \operatorname{Res}_{z=z_0} f(z)$.

γ: a contour around z_0

Cauchy residue thm: If γ is a positively oriented simple closed contour, and $f(z)$ is analytic everywhere in and on γ except for a finite number of isolated singular points $\{z_n\}$ inside γ, then: $\int_\gamma f(z)\, dz = 2\pi i \sum_{r=1}^{n} \operatorname{Res}_{z=z_r} f(z)$

Finding the residue:

Method 1: If $f(z) = \frac{\phi(z)}{z-z_0}$ near z_0 and $\phi(z)$ analytic at z_0 then $\operatorname{Res}_{z=z_0} f(z) = \phi(z_0)$

Method 2: If $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ near z_0 and $\phi(z)$ analytic at z_0 and $m \ge 1$ then $\operatorname{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$

Method 3: If $p(z), q(z)$ analytic at z_0 and $q(z)$ has a simple zero at z_0 (i.e. $q(z_0)=0$ and $q'(z_0) \ne 0$) then $\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$

Classification of isolated singular points: If z_0 is an isolated singular point of f, then f has a Laurent series in $\operatorname{Ann}(z_0, 0, r)$ $(r>0)$.

Look at the principal part of the Laurent series:
- If all b_i are zero, then the singular point is removable
- If only finitely many b_i are nonzero, then the singular point is a pole
- Otherwise, the singular point is essential

Essential singular point (Picard thm): $\longrightarrow$ the neighbourhood of z_0 (excluding z_0 itself) attains any value in f except possibly one point

(i.e. $\forall r>0$, $\{f(z) \mid z \in \operatorname{Ann}(z_0,0,r)\} = \mathbb{C} \setminus \{z_1\}$ for some $z_1 \in \mathbb{C}$)

Removable singular point: If we replace $f(z_0)$ with $a_0 = \lim_{z\to z_0}$ (<Laurent series around z_0>), then the resulting function is analytic at z_0.

Pole singular point: ① $f(z) = \frac{\phi(z)}{(z-z_0)^n}$ for some $\phi(z)$ analytic at a neighbourhood of z_0 where $\phi(z_0) \ne 0$ and $n \in \mathbb{N}$

② $\lim_{z\to z_0} f(z) = \infty$

- **Order of a pole:** the largest $n \in \mathbb{N}$ s.t. $b_n \ne 0$ (simple pole: $n=1$; double pole: $n=2$)

(if f has a pole of order n, then $\frac{1}{f}$ has a zero of order n)

Order of a zero: If $f(z)$ is analytic at z_0 (so $f(z)$ has a Taylor series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$) then the order of zero at z_0 is the smallest n s.t. $a_n \ne 0$.

(the order of zero at z_0 is at least one $\iff f(z_0) = 0$)

(the order of zero at z_0 is $n \iff f(z) = (z-z_0)^n g(z)$ where $g(z_0) \ne 0$)

Thm: If p, q are analytic around z_0 and the order of zero at z_0 for $p(z)$ is n and the order of zero at z_0 for $q(z)$ is m then:

① z_0 is an isolated singular point of $\frac{p(z)}{q(z)}$.

② If $m > n$ then z_0 is a pole with order $m-n$, otherwise z_0 is removable.